# UNIVERSAL FAMILIES ON MODULI SPACES OF PRINCIPAL BUNDLES ON CURVES

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ABSTRACT. Let $H$ be a connected semisimple linear algebraic group defined over $\mathbb{C}$ and $X$ a compact connected Riemann surface of genus at least three. Let $\mathcal{M}'_X(H)$ be the moduli space parametrising all topologically trivial stable principal $H$-bundles over $X$ whose automorphism group coincides with the centre of $H$. It is a Zariski open dense subset of the moduli space of stable principal $H$-bundles. We prove that there is a universal principal $H$-bundle over $X \times \mathcal{M}'_X(H)$ if and only if $H$ is an adjoint group (that is, the centre of $H$ is trivial).

## 1. INTRODUCTION

Let $X$ be a compact connected Riemann surface of genus at least three. Let $\mathcal{M}_X(n, d)$ denote the moduli space of all stable vector bundles over $X$ of rank $n$ and degree $d$, which is a smooth irreducible quasiprojective variety defined over $\mathbb{C}$. A vector bundle $\mathcal{E}$ over $X \times \mathcal{M}_X(n, d)$ is called *universal* if for every point $m \in \mathcal{M}_X(n, d)$, the restriction of $\mathcal{E}$ to $X \times \{m\}$ is in the isomorphism class of holomorphic vector bundles over $X$ defined by $m$. A well-known theorem says that there is a universal vector bundle over $X \times \mathcal{M}_X(n, d)$ if and only if $d$ is coprime to $n$ (see [13] for existence in the coprime case, [8] for non-existence in the non-coprime case and [7] for a topological version of non-existence in the case $d = 0$).

Let $H$ be a connected semisimple linear algebraic group defined over the field of complex numbers. Ramanathan extended the notion of (semi)stability to principal $H$-bundles and constructed moduli spaces for stable principal $H$-bundles over $X$ [9, 10, 11]. The construction works for any given topological type, yielding a moduli space which is an irreducible quasiprojective variety defined over $\mathbb{C}$. We are concerned here with the case of topologically trivial stable principal $H$-bundles. Let $\mathcal{M}_X(H)$ denote the moduli space of topologically trivial stable principal $H$-bundles over $X$.

Let $Z(H) \subset H$ be the centre. For any $H$-bundle $E_H$, the group $Z(H)$ is contained in the automorphism group $\mathrm{Aut}(E_H)$. Let

$$\mathcal{M}'_X(H) \subset \mathcal{M}_X(H)$$

be the subvariety consisting of all $H$-bundles $E_H$ over $X$ with the property $\mathrm{Aut}(E_H) \cong Z(H)$. It is known that $\mathcal{M}'_X(H)$ is a dense Zariski-open subset contained in the smooth locus of $\mathcal{M}_X(H)$.

A principal $H$-bundle $E$ over $X \times \mathcal{M}'_X(H)$ will be called a *universal bundle* if for every point $m \in \mathcal{M}_X(n, d)$, the restriction of $\mathcal{E}$ to $X \times \{m\}$ is in the isomorphism class of stable $H$-bundles over $X$ defined by the point $m$ of the moduli space.

The following theorem is the main result proved here:

**Theorem 1.1.** *There is a universal $H$-bundle over $X \times \mathcal{M}'_X(H)$ if and only if $Z(H) = e$.*

**Remark 1.2.** Non-existence for $H = \mathrm{SL}(n, \mathbb{C})$ was previously known [8, 7], as also was existence for $H = \mathrm{PGL}(n, \mathbb{C})$.

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## 2. EXISTENCE OF UNIVERSAL BUNDLE

We begin this section by recalling very briefly certain facts from [2]. Unless otherwise specified, all bundles and sections considered will be algebraic.

**Remark 2.1.** Let $E$ be a principal $G$-bundle over $X$, where $G$ is a reductive linear algebraic group defined over $\mathbb{C}$, and $H \subset G$ is a Zariski closed semisimple subgroup. For any variety $Y$ equipped with an action of $G$, the fibre bundle $(E \times Y)/G$ over $X$ associated to $E$ will be denoted by $E(Y)$.

(1) There is a natural action of the group $\mathrm{Aut}\,_G E$, defined by all automorphisms of $E$ over the identity map of $X$ that commute with the action of $G$, on $\Gamma(X, E(G/H))$ (the space of all holomorphic sections of the fibre bundle $E(G/H) = E/H$ over $X$) and the orbits correspond to the equivalence classes of $H$-reductions of $E$ with two reductions being equivalent if the corresponding principal $H$-bundles are isomorphic.

(2) Let $G = \mathrm{GL}(n, \mathbb{C})$, and let $\phi : H \hookrightarrow G$ be a faithful representation of the semisimple group $H$. Let $Q$ denote the open subset of semistable principal $G$-bundles (or equivalently, of trivial determinant semistable vector bundles of rank $n$) of the usual "Quot scheme", and let $Q(\phi)$ be the "Quot scheme" which parametrises pairs of the form $(E', s)$, where $E'$ is a principal $G$-bundle and $s$ is a reduction of structure group of $E'$ to $H$. Then $Q(\phi)$ is in a sense a "relative Quot scheme". As is clear from the definition and the notation, this scheme is dependent on the choice of the inclusion $\phi : H \hookrightarrow G$ (for details see [2, 10, 11]).

One also has a *tautological* sheaf on $X \times Q$ which in fact is a vector bundle. We denote by $\mathcal{E}$ the *associated tautological* principal $G$-bundle on $X \times Q$.

Recall that the moduli space of principal $G$-bundles ($G = \mathrm{GL}(n, \mathbb{C})$) is realised as a good quotient of $Q$ by the action of a reductive group $\mathcal{G}$. We may also assume that the group $\mathcal{G}$ is with trivial centre (see for example [4]).

**Remark 2.2.** It is immediate that the action of $\mathcal{G}$ on $Q$ lifts to an action on $Q(\phi)$, where $\mathcal{G}$, $Q$ and $Q(\phi)$ are defined in Remark 2.1.

We have a morphism

$$(1) \qquad\qquad\qquad \psi \,:\, Q(\phi) \longrightarrow Q\,,$$

which sends any $H$-bundle $E'$ to the $\mathrm{GL}(n, \mathbb{C})$-bundle obtained by extending the structure group of $E'$ using the homomorphism $\phi$. In fact, $\psi$ is a $\mathcal{G}$-equivariant *affine morphism* (see [2]).

Continuing with the notation in the above two remarks, consider the $\mathcal{G}$-action on $Q(\phi)$ (defined in Remark 2.1(2)) with the linearisation induced by the affine $\mathcal{G}$-morphism $\psi$ in (1). Since a good quotient of $Q$ by $\mathcal{G}$ exists and since $\psi$ is an affine $\mathcal{G}$-equivariant map, a good quotient $Q(\phi)//\mathcal{G}$ exists (see [11, Lemma 5.1]).

Moreover by the universal property of categorical quotients, the canonical morphism

(2) $$\overline{\psi} : Q(\phi)//\mathcal{G} \longrightarrow Q//\mathcal{G}$$

given by $\psi$ is also *affine*.

**Theorem 2.3.** *Let $\mathcal{M}_X(H)$ denote the scheme $Q(\phi)//\mathcal{G}$ (see (2)). Then this scheme is the coarse moduli scheme of semistable $H$-bundles. Further, the scheme $\mathcal{M}_X(H)$ is projective, and if $H \hookrightarrow \mathrm{GL}(V)$ is a faithful representation, then the canonical morphism*

$$\overline{\psi} : \mathcal{M}_X(H) \longrightarrow \mathcal{M}_X(\mathrm{GL}(V)) = Q//\mathcal{G}$$

*is finite.*

Let $Q(\phi)^s$ be the open subscheme of $Q(\phi)$ consisting of stable $H$-bundles.

**Lemma 2.4.** *Let $Q'_H \subset Q(\phi)^s$ be the subset parametrising all stable $H$-bundles whose automorphism group is $Z(H)$. Then the action of $\mathcal{G}$ on the subset $Q'_H$ is free, and furthermore, the quotient morphism $Q'_H \longrightarrow Q'_H/\mathcal{G}$ is a principal $\mathcal{G}$-bundle. In fact, $Q'_H/\mathcal{G}$ is precisely the Zariski open subset $\mathcal{M}'_X(H)$ (the variety $\mathcal{M}'_X(H)$ is defined in the introduction).*

*Proof.* For any point $E_H \in Q(\phi)$, the isotropy subgroup of $E_H$ for the action of $\mathcal{G}$ coincides with $\mathrm{Aut}(E_H)/Z(H)$. This can be seen as follows: firstly, the point $E_H$ is a pair $(E, s)$, where $E \in Q$ and $s$ is a reduction of structure group to $H$ of the $G$-bundle $E$. It is well known for the action of $\mathcal{G}$ on $Q$ that the isotropy at $E$ is precisely the group $\mathrm{Aut}(E)/Z(G)$. From this it is easy to see that the isotropy of $E_H$ for the action of $\mathcal{G}$ on $Q(\phi)$ is the group $\mathrm{Aut}(E, s)/Z(H) = \mathrm{Aut}(E_H)/Z(H)$.

Hence it follows that the action of $\mathcal{G}$ on the open subset $Q'_H$ is free, and the proof of the lemma is complete.  □

We remark that, at least when $H$ is of adjoint type, $Q'_H$ is a non-empty open subset of $Q(\phi)^s$. Openness is easy and can be seen for example from [5, Theorem II.6 (ii)]. Non-emptiness follows from Proposition 2.6 below.

We prove a proposition on semisimple groups, possibly known to experts but which we could not locate in any standard text.

**Proposition 2.5.** *Let $H$ be a semisimple algebraic group. Then $H$ has a faithful, irreducible representation $\phi : H \longrightarrow \mathrm{GL}(V)$ if and only if the centre of $H$ is cyclic.*

*Proof.* One way the implication is easy, namely suppose that a faithful irreducible representation exists, then the centre $Z(H)$ of $H$ is cyclic. To see this, first note that under the representation $\phi$, the centre $Z(H)$ maps to a subgroup which commutes with all elements of $\phi(H)$. Since $\phi$ is irreducible, this implies by Schur's Lemma that $\phi(Z(H)) \subset Z(\mathrm{GL}(V))$, where $Z(\mathrm{GL}(V))$ is the centre of $\mathrm{GL}(V)$. Observe further that since $H$ is semisimple, we have $\phi(H) \subset \mathrm{SL}(V)$. Hence, $Z(H) \subset Z(\mathrm{SL}(V))$ and is therefore cyclic.

The contrapositive statement is harder to prove. We proceed as follows: Let $H'$ be the simply connected cover of $H$, and let $\overline{H}$ be the associated adjoint group, namely $H/Z(H)$. Let $\Lambda$ (respectively, $\Lambda_R$) be the weight lattice (respectively, root lattice). In other words, by the Borel-Weil theorem $\Lambda = \mathcal{X}(B)$ and $\Lambda_R = \mathcal{X}(\overline{B})$, where $B$, $\overline{B}$ are fixed Borel subgroups of $H$, $\overline{H}$. Then, from the exact sequence

$$e \longrightarrow Z(H) \longrightarrow H \longrightarrow \overline{H} \longrightarrow e$$

we see that $\Lambda/\Lambda_R \simeq \mathcal{X}(Z(H))$. In other words, the quotient group $\Lambda/\Lambda_R$ is cyclic of order $m$.

Let $\overline{\lambda}$ be a generator of the cyclic group $\Lambda/\Lambda_R$. Then, by an action of the Weyl group we may assume that the coset representative $\lambda \in \Lambda$ is actually a dominant weight.

Suppose that the root lattice has the following decomposition (corresponding to the simple components of $\overline{H}$):

$$\Lambda_R = \bigoplus_{i=1}^{\ell} \Lambda^i .$$

Then, by possibly adding dominant weights from the $\Lambda^i$, we may assume that $(m \cdot \lambda) \in \Lambda_R$ has all its direct sum components $\lambda_i \neq 0$, $i \in [1, \ell]$, where $\lambda$ is as above.

For this choice of $\lambda \in \Lambda$ let $V_\lambda$ be the corresponding $H$-module given by

$$\phi_\lambda : H \longrightarrow \mathrm{GL}(V_\lambda) .$$

Then one knows that $\phi_\lambda$ is an irreducible representation of $H$.

We claim that the representation $\phi_\lambda$ is even *faithful*. Suppose that this is not the case. Let $K_\lambda := \mathrm{kernel}(\phi_\lambda) \neq e$.

Firstly, $K_\lambda \subset Z(H)$. To see this, let $K'$ be the inverse image of $K_\lambda$ in the simply connected cover $H'$ of $H$. Then the choice of $\lambda$ so made that its simple components are non-zero in fact forces the following: Suppose that $H' = H_1 \times \cdots \times H_\ell$ is the decomposition of $H'$ into its almost simple factors. (We recall that a semisimple algebraic group is called almost simple if the quotient of it by its centre is simple.) Then the normal subgroup $K'$ in its decomposition in $H'$ is such that $K_i := K' \bigcap H_i$ are proper normal subgroups of $H_i$. In particular, $K_i \subset Z(H_i)$ for all $i \in [1, \ell]$. This implies that $K' \subset Z(H')$ and hence, $K_\lambda \subset Z(H)$.

Note that the dominant character $\lambda$ is non-trivial on the generator of the centre $Z(H)$ because $\mathcal{X}(Z(H)) = \Lambda/\Lambda_R$. Now $K_\lambda \subset Z(H)$ and $Z(H)$ cyclic implies that $\lambda$ is non-trivial on the generator of $K_\lambda$ as well. This contradicts the fact that $K_\lambda = \mathrm{kernel}(\phi_\lambda)$. This proves the claim. Therefore, the proof of the proposition is complete. $\qquad\square$

If $H$ is of adjoint type, then by Proposition 2.5 we can choose the inclusion $\phi : H \hookrightarrow G$ in Remark 2.1(2) to be an irreducible representation. Henceforth, $\phi$ will be assumed to be irreducible.

Let $E$ be a stable $H$-bundle of trivial topological type. Recall that one can realise $E$ from a unique, up to an inner conjugation, irreducible representation of $\pi_1(X)$ in a maximal compact subgroup of $H$ (see [9]). For notational convenience, we will always suppress the base point in the notation of fundamental group. Denote by $M(E)$ the Zariski closure of the image of $\pi_1(X)$ in $H$.

**Proposition 2.6.** *Let $H$ be of adjoint type and let $\phi$ be a faithful irreducible representation of $H$ in $V$ (see Proposition 2.5). Let $U \subset Q(\phi)^s$ be the subset defined as follows:*

$$U = \{E \in Q(\phi)^s \,|\, M(E) = H\}.$$

*Then $U$ is non-empty and is contained in the subset $Q'_H$ of stable $H$-bundles which have trivial automorphism group. (Since $H$ is of adjoint type, $Z(H) = e$.)*

*Proof.* Since $\phi$ is irreducible, using Lemma 2.1 of [12] we conclude that there is an irreducible representation

$$\rho \,:\, \pi_1(X) \longrightarrow H$$

which have the property that the composition $\phi \circ \rho : \pi_1(X) \longrightarrow G := \mathrm{GL}(V)$ continues to remain irreducible. This implies that $M(E_\rho) = H$ by the construction in [12, Lemma 2.1] and hence $E_\rho \in U$, i.e., $U$ is non-empty.

Let $E \in U$. Then by the definition of $U$, there exists a representation $\rho$ of $\pi_1(X)$ in $H$ such that $E \simeq E_\rho$, where $E_\rho$ is the flat principal $H$-bundle given by $\rho$. Observe that $E_\rho$ is a stable $H$-bundle and the associated $G$-bundle is also stable. Hence all the automorphisms of this associated $G$-bundle lie in $Z(G)$. Since $H$ is of adjoint type, it follows that the $H$-bundle $E_\rho$ has no non-trivial automorphisms. Hence it follows that $U \subset Q'_H$. $\qquad\square$

We have the following theorem on existence of universal families.

**Theorem 2.7.** *Let $H$ be a group of adjoint type and $\mathcal{M}'_X(H)$ the Zariski open subset of $\mathcal{M}_X(H)$ defined in the introduction. Then there exists a universal family of principal $H$-bundles on $X \times \mathcal{M}'_X(H)$.*

*Proof.* We first observe that the variety $\mathcal{M}'_X(H)$ is precisely the image of $Q'_H$ under the quotient map for the action of $\mathcal{G}$. We recall that $Q'_H$ is nonempty by Proposition 2.6.

Since $Z(H) = e$, it follows that $H \subset \overline{G} := G/Z(G)$, where $G = \mathrm{GL}(V)$ is as in Remark 2.1(2).

Consider the tautological $G$-bundle $\mathcal{E}$ on $X \times Q^s$, and let $\overline{\mathcal{E}}$ be the corresponding $\overline{G}$-bundle obtained by extending the structure group. Then it is well-known that the adjoint universal bundle $\overline{\mathcal{E}}$ descends to the quotient $\mathcal{M}_X(G)^s$ (see [4]).

We follow the same strategy for $\mathcal{M}'_X(H)$ as well. Consider the pulled back $\overline{G}$-bundle $(\mathrm{Id}_X \times \psi)^*\overline{\mathcal{E}}$ on $X \times Q'_H$, where $\psi$ is the map in (1) and $\overline{G} := G/Z(G)$. The action of $\mathcal{G}$ on $Q'_H$ is free by Lemma 2.4. Therefore, the quotient $Q'_H \longrightarrow \mathcal{M}'_X(H)$ is a principal $\mathcal{G}$-bundle. Further, the action of $\mathcal{G}$ lifts to the tautological bundle $(\mathrm{Id}_X \times \psi)^*\overline{\mathcal{E}}$. In particular, the principal $\overline{G}$-bundle $(\mathrm{Id}_X \times \psi)^*\overline{\mathcal{E}}$ descends to a principal $\overline{G}$-bundle over $X \times \mathcal{M}'_X(H)$.

Let us denote this descended $\overline{G}$-bundle over $X \times \mathcal{M}'_X(H)$ by $\overline{\mathcal{E}}_0$.

Let $\pi \,:\, (\mathrm{Id}_X \times \psi)^*\mathcal{E}(G/H) \longrightarrow (\mathrm{Id}_X \times \psi)^*\overline{\mathcal{E}}(\overline{G}/H)$ be the natural map induced by the projection $G/H \longrightarrow \overline{G}/H$, where $\psi$ is the map in (1). We note that the universal $H$-bundle over $X \times U$, where $U$ is defined in Proposition 2.6, is a reduction of structure group of the pulled back $G$-bundle $(\mathrm{Id}_X \times \psi)^*\mathcal{E}$. Let

$$\sigma \,:\, X \times U \longrightarrow (\mathrm{Id}_X \times \psi)^*\mathcal{E}(G/H)$$

be the section giving this reduction of structure group. Then the composition $\pi \circ \sigma$ is a section of $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(\overline{G}/H)$ over $X \times U$.

Since $H$ is semisimple, a lemma of Chevalley says that there is a $\overline{G}$-module $W$ and an element $w \in W$ such that $H$ is precisely the isotropy subgroup for $w$ (see [3, p. 89, Theorem 5.1]). Therefore, $\overline{G}/H$ is identified with the closed $\overline{G}$-orbit in $W$ defined by $w$. Then we see that $\overline{\mathcal{E}}(\overline{G}/H) \hookrightarrow \overline{\mathcal{E}}(W)$. We may therefore view the section $\pi \circ \sigma : X \times U \longrightarrow (\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(\overline{G}/H)$ as a section of the vector bundle $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(W)$ over $X \times U$.

Since $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}$ descends to the $\overline{G}$-bundle $\overline{\mathcal{E}}_0$ over $X \times \mathcal{M}'_X(H)$, it follows that the associated vector bundle $(\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(W)$ also descends to $X \times \mathcal{M}'_X(H)$. Clearly this vector bundle is nothing but the associated vector bundle $\overline{\mathcal{E}}_0(W)$, associated to the $\overline{G}$-bundle $\overline{\mathcal{E}}_0$ on $X \times \mathcal{M}'_X(H)$ for the $\overline{G}$-module $W$.

Since each point of $\mathcal{M}'_X(H)$ represents an isomorphism class of stable $H$-bundle, it follows that set theoretically the reduction section $\pi \circ \sigma \in \Gamma((\mathrm{Id}_X \times \psi)^* \overline{\mathcal{E}}(W))$ descends to a section on $X \times \mathcal{M}'_X(H)$ of the descended vector bundle $\overline{\mathcal{E}}_0(W)$.

We now appeal to [6, Proposition 4.1], which implies that the section $\pi \circ \sigma$ in fact descends to give a holomorphic section of $\overline{\mathcal{E}}_0(W)$ over $X \times \mathcal{M}'_X(H)$.

Again set-theoretically, the image of this section of $\overline{\mathcal{E}}_0(W)$ lies in $\overline{\mathcal{E}}_0(\overline{G}/H) \subset \overline{\mathcal{E}}_0(W)$. As before, from [6, Proposition 4.1] it follows that $\pi \circ \sigma$ gives a reduction of structure group to $H$ of the descended $\overline{G}$-bundle $\overline{\mathcal{E}}_0$. The $H$-bundle over $X \times \mathcal{M}'_X(H)$ obtained this way is the required universal $H$-bundle. This completes the proof of the theorem.    □

## 3. Nonexistence of universal bundle

Let $H$ be a complex semisimple linear algebraic group, and let $K \subset H$ be a maximal compact subgroup. The Lie algebra of $K$ will be denoted by $\mathfrak{k}$. A homomorphism

$$\rho : \pi_1(X) \longrightarrow K$$

is called *irreducible* if no nonzero vector in $\mathfrak{k}$ is fixed by the adjoint action of the subgroup $\rho(\pi_1(X)) \subset K$ on $\mathfrak{k}$. Let $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$ denote the space of all irreducible homomorphisms such that the corresponding $K$-bundle is topologically trivial. So any homomorphism in $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$ is induced by a homomorphism from $\pi_1(X)$ to the universal cover of $K$.

Let $g$ denote the genus of $X$. Assume that $g \geq 3$. If we choose a basis

$$\{a_1, \cdots, a_g, b_1, \cdots, b_g\} \subset \pi_1(X)$$

such that

$$(3) \qquad \pi_1(X) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g : \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} \rangle,$$

then $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$ gets identified with a real analytic subspace of $K^{2g}$.

For any $\rho \in \mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)$, the principal $H$-bundle obtained by extending the structure group of the principal $K$-bundle given by $\rho$ is stable. A theorem of Ramanathan says that all topologically trivial stable $H$-bundles arise in this way, that is, the space of all equivalence classes of irreducible homomorphisms of $\pi_1(X)$ to $K$ is in bijective correspondence with the space of all stable $H$-bundles over $X$ [9, Theorem 7.1]. More precisely

(see the proof of [9, Theorem 7.1]), the real analytic space underlying the moduli space $\mathcal{M}_X(H)$ parametrising topologically trivial stable $H$-bundles is analytically isomorphic to the quotient space

$$R(\pi_1(X), H) := \mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K)/K,$$

for the action constructed using the conjugation action of $K$ on itself. Let

(4)
$$q : \mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), K) \longrightarrow R(\pi_1(X), H) \cong \mathcal{M}_X(H)$$

be the quotient map. The open subset $\mathcal{M}'_X(H) \subset \mathcal{M}_X(H)$ is a smooth submanifold and the restriction of the map in (4)

(5)
$$q|_{q^{-1}(\mathcal{M}'_X(H))} : q^{-1}(\mathcal{M}'_X(H)) \longrightarrow \mathcal{M}'_X(H)$$

is a smooth principal $K/Z$-bundle, where $Z$ is the centre of $K$. Note that $Z$ coincides with the centre of $H$ (as $H$ is semisimple, its centre is a finite group).

If $\mathcal{U}_H$ is a universal $H$-bundle over $X \times \mathcal{M}'_X(H)$, then we have a reduction of structure group of $\mathcal{U}_H$ to $K$ which is constructed using the correspondence established in [9] between stable $H$-bundles and irreducible flat $K$-bundles. Let

(6)
$$\mathcal{U}_K \longrightarrow X \times \mathcal{M}'_X(H)$$

be the smooth principal $K$-bundle obtained from $\mathcal{U}_H$ this way. Consider the principal $K/Z$-bundle over $X \times \mathcal{M}'_X(H)$, where $Z \subset K$ is the centre, obtained by extending the structure group of $\mathcal{U}_K$ using the natural projection of $K$ to $K/Z$. The restriction of this $K/Z$-bundle to $p \times \mathcal{M}'_X(H)$, where $p$ is the base point in $X$ used for defining $\pi_1(X)$, is identified with the $K/Z$-bundle in (5). To see this first note that the universal $K$-bundle over $X \times \mathrm{Hom}(\pi_1(X), K)$ is obtained as a quotient by the action of $\pi_1(X)$ on $\widetilde{X} \times \mathrm{Hom}(\pi_1(X), K) \times K$, where $\widetilde{X}$ is the pointed universal cover of $X$ for the base point $p$; the action of $z \in \pi_1(X)$ sends any $(\alpha, \beta, \gamma)$ to $(\alpha, z, \beta, \beta(z)^{-1}\gamma)$. From this it follows that the restriction of this universal bundle to $p \times \mathrm{Hom}(\pi_1(X), K)$ is canonically trivialized. The above mentioned identification is constructed using this trivialization.

Our aim is to show that no $K$-bundle over $\mathcal{M}'_X(H)$ exists that produces the $K/Z$-bundle in (5) by extension of structure group, provided the centre $Z$ is non-trivial.

Let $F_3$ denote the free group on three generators. Fix a surjective homomorphism

(7)
$$f : \pi_1(X) \longrightarrow F_3$$

that sends $a_i$, $1 \leq i \leq 3$, to the $i$-th generator of $F_3$ and sends $a_i$, $4 \leq i \leq g$, and $b_i$, $1 \leq i \leq g$, to the identity element, where $a_j$, $b_j$ are as in (3) (recall that $g \geq 3$).

Set

(8)
$$R(F_3, H) := \mathrm{Hom}^{\mathrm{irr}}(F_3, K)/K$$

to be the equivalence classes of irreducible representations. Note that $\mathrm{Hom}(F_3, K) \simeq K^3$, and under this identification the action

$$\mu : \mathrm{Hom}(F_3, K) \times K \longrightarrow \mathrm{Hom}(F_3, K)$$

given by $\mu(\rho, A) = A^{-1}\rho A$ corresponds to the simultaneous diagonal conjugation action of $K$ on the three factors.

Since $F_3$ is a free group and $K$ is connected, any homomorphism $\rho$ from $F_3$ to $K$ can be deformed to the trivial homomorphism. This implies that the principal $H$-bundle corresponding to the homomorphism

$$\rho \circ f \, : \, \pi_1(X) \longrightarrow K$$

is topologically trivial, where $f$ is defined in (7). Therefore, we have an embedding

$$(9) \qquad\qquad R(F_3, H) \longrightarrow R(\pi_1(X), H) \cong \mathcal{M}_X(H)$$

that sends any $\rho$ to $\rho \circ f$. Let $R'(F_3, H) \subset R(F_3, H)$ be the inverse image of the open subset $\mathcal{M}'_X(H)$ under the above map. By Proposition 2.6 it follows that $R'(F_3, H)$ is a *non-empty* open subset of $R(F_3, H)$.

Fix a point $p \in X$, which will also be the base point for the fundamental group. Consider the restriction of the principal $K$-bundle $\mathcal{U}_K$ in (6) to $p \times \mathcal{M}'_X(H) \hookrightarrow X \times \mathcal{M}'_X(H)$ and denote it by $\mathcal{U}_{K,p}$.

Let

$$(10) \qquad\qquad \gamma \, : \, p \times R'(F_3, H)) \longrightarrow p \times \mathcal{M}'_X(H)$$

be the map given by the embedding $R'(F_3, H) \longrightarrow \mathcal{M}'_X(H)$ constructed in (9). Taking the pull back of the above defined principal $K$-bundle $\mathcal{U}_{K,p}$ over $p \times \mathcal{M}'_X(H)$ under the morphism $\gamma$ in (10),

$$(11) \qquad
\begin{array}{ccc}
\widetilde{M} & & \mathcal{U}_{K,p} \\
\downarrow & & \downarrow \\
p \times R'(F_3, H)) & \xrightarrow{\ \gamma\ } & p \times \mathcal{M}'_X(H)
\end{array}$$

we obtain a principal $K$-bundle $\widetilde{M} \, := \, \gamma^* \mathcal{U}_{K,p}$ on $R'(F_3, H)$ whose associated $K/Z$-bundle (as before, $Z$ is the centre of $K$) is precisely the $K/Z$-bundle $q^{-1}(R'(F_3, H)) \longrightarrow R'(F_3, H)$ constructed in (5). In particular, $q^{-1}(R'(F_3, H)) \simeq \widetilde{M}/Z$.

Let $\mathrm{Hom}^{\mathrm{irr}}(F_3, K)' \subset \mathrm{Hom}^{\mathrm{irr}}(F_3, K)$ be the inverse image of $R'(F_3, H)$ under the quotient map in (8). Let

$$(12) \qquad\qquad q_0 \, : \, \mathrm{Hom}^{\mathrm{irr}}(F_3, K)' \longrightarrow R'(F_3, H)$$

be the restriction of the quotient map in (8). So $q_0$ is the quotient for the conjugation action of $K$. This map $q_0$ defines a principal $K/Z$-bundle over $R'(F_3, H)$ which is evidently identified with the $K/Z$-bundle $\widetilde{M}/Z \longrightarrow R'(F_3, H)$ obtained from (11).

We will prove (by contradiction) that such a $K$-bundle $\widetilde{M}$ does not exist. In other words, we will show that there is no principal $K$-bundle over $R'(F_3, H)$ whose extension of structure group is the $K/Z$-bundle in (12).

Assume the contrary, then we get the following diagram of topological spaces and morphisms:

$$
(13) \quad
\begin{array}{ccccc}
Z & \longrightarrow & K & \longrightarrow & K/Z \\
\| & & \downarrow & & \downarrow \\
Z & \longrightarrow & \widetilde{M} & \longrightarrow & q_0^{-1}(R'(F_3, H)) \\
& & \downarrow & & \downarrow \\
& & R'(F_3, H) & = & R'(F_3, H)
\end{array}
$$

We will need a couple of results. The proof of the non-existence of the bundle $\widetilde{M}$ will be completed after establishing Proposition 3.2.

**Lemma 3.1.** *There exist finitely many compact differentiable manifolds $N_i$ and differentiable maps*

$$
f_i \; : \; N_i \; \longrightarrow \; K^3 \, ,
$$

*$1 \leq i \leq d$, such that if $N := \bigcup_{i=1}^{d} f_i(N_i)$ then*

  (1) *the complement $K^3 \setminus N$ is contained in $q_0^{-1}(R'(F_3, H))$, where $q_0$ is the projection in (12).*
  (2) *Furthermore, $\dim K^3 - \dim N_i \geq 4$ for all $i \in [1, d]$.*

*Proof.* Take any homomorphism $\rho \; : \; \pi_1(X) \; \longrightarrow \; K$. Let $E_\rho$ denote the corresponding polystable principal $H$-bundle over $X$ [9]. The automorphism group of the polystable $H$-bundle $E_\rho$ coincides with the centraliser of $\rho(\pi_1(X))$ in $H$. Hence if $\rho(\pi_1(X))$ is dense in $K$, then the automorphism group of $E_\rho$ is the centre $Z(H) \subset H$. Therefore, if the topological closure $\overline{\rho(\pi_1(X))}$ of $\rho(\pi_1(X))$ is $K$, and $E_\rho$ is topologically trivial, then $E_\rho \in \mathcal{M}'_X(H)$.

Now assume that the centraliser $C(\rho) \subset H$ of $\rho(\pi_1(X))$ in $H$ properly contains $Z(H)$. Since the complexification of $\overline{\rho(\pi_1(X))}$ is reductive, and the centraliser of a reductive group is reductive, we conclude that $C(\rho)$ is reductive. Take a semisimple element $z \in (H \setminus Z(H)) \bigcap C(\rho)$ (since $C(\rho)$ is reductive and larger than $Z(H)$ such an element exists). Let $C_z \subset K$ be the centraliser of $z$ in $K$. We have $\overline{\rho(\pi_1(X))} \subset C_z$, and $C_z$ is a proper subgroup of $K$ as $z \notin Z(H)$. Also, $C_z$ contains a maximal torus of $K$.

Fix a maximal torus $T \subset K$. (Since any two maximal tori are conjugate, any subgroup $C_z$ of the above type would contain $T$ after an inner conjugation.) Consider all proper Lie subgroups $M$ of $K$ satisfying the following two conditions:

  (i) $T \subseteq M$, and
  (ii) there exists a semisimple element $z \in H \setminus Z(H)$ such that $M$ is the centraliser of $z$ in $K$.

Let $\mathcal{S}$ denote this collection of Lie subgroups of $K$.

The connected ones among $\mathcal{S}$ are precisely the maximal compact subgroups of Levi subgroups of proper parabolic subgroups of $H$ containing $T$. Note that if $P$ is a proper parabolic subgroup of $H$, then $P \bigcap K$ is a maximal compact subgroup of a Levi subgroup of $P$. All the connected ones among $\mathcal{S}$ arise as $P \bigcap K$ for some proper parabolic subgroup $P \subset H$ containing $T$.

This collection $\mathcal{S}$ is a finite set. To see this, we first note that there are only finitely many parabolic subgroups of $H$ that contain $T$. For a proper parabolic subgroup $P \subset H$ containing $T$, there are only finitely many Lie subgroups of $K$ that have $P \bigcap K$ as the connected component containing the identity element. Thus $\mathcal{S}$ is a finite set.

Let $M_1, \cdots, M_d$ be the subgroups of $K$ that occur in $\mathcal{S}$.

We will show that the codimension of each $M_i$ in $K$ is at least two. It suffices show that for a maximal proper parabolic subgroup $P$ of $G$ containing $T$, the codimension of $M := P \bigcap K$ in $K$ is at least two.

To prove that the codimension of $M = P \bigcap K$ in $K$ is at least two, let $\mathfrak{k}$ be the Lie algebra of $K$, and let $\mathfrak{h}$ be the Lie algebra of $T$ (the Cartan subalgebra). Let

$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha \text{ a root}} \mathfrak{k}^{\alpha}$$

be the root space decomposition of $\mathfrak{k}$. Let $\mathfrak{m}$ be a Lie algebra of $M$ which is a Lie subalgebra of $\mathfrak{k}$ containing $\mathfrak{h}$. Then we have the decomposition

$$\mathfrak{m} = \mathfrak{h} + \sum_{\alpha \text{ a root}} \mathfrak{m}^{\alpha}$$

for $\mathfrak{m}$ where each $\mathfrak{m}^{\alpha}$ is irreducible for $\mathfrak{h}$, and hence has to coincide with one of the $\mathfrak{k}^{\alpha}$. Then $\mathrm{codim}_{\mathfrak{k}}(\mathfrak{m}) \geq 2$ since if $\alpha$ is a root for the subalgebra $\mathfrak{m}$ then so is $-\alpha$ (see [1, p. 83, Corollary 4.15]).

Consequently, the codimension of $M$ in $K$ is at least two. Thus the codimension of each $M_i$, $i \in [1, d]$, in $K$ is at least two.

For each $i \in [1, d]$, let

(14)
$$\overline{f}_i : K \times M_i^3 \longrightarrow K^3$$

be the map defined by $(x, (y_1, y_2, y_3)) \longmapsto (xy_1x^{-1}, xy_wx^{-1}, xy_3x^{-1})$. Consider the free action of $M_i$ on $K \times M_i^3$ defined by

$$z \cdot (x, (y_1, y_2, y_3)) = (xz^{-1}, (zy_1z^{-1}, zy_2z^{-1}, zy_3z^{-1})),$$

where $x \in K$ and $z, y_1, y_2, y_3 \in M_i$. The map $f_i$ in (14) clearly factors through the quotient

$$\frac{K \times M_i^t}{M_i}$$

for the above action. Therefore, we have

(15)
$$f_i : N_i := \frac{K \times M_i^t}{M_i} \longrightarrow K^t$$

induced by $f_i$.

To prove part (1) of the lemma, we recall the earlier remark that for any homomorphism $\rho' \in q_0^{-1}(R'(F_3, H))$ (the map $q_0$ is defined in (12)), the automorphism group of the principal $H$-bundle corresponding to $\rho' \circ f$ (the homomorphism $f$ is defined in (7)) coincides with $Z(H)$ if the image of $\rho'$ is dense in $K$. From the properties of the collection

$\{M_1, \cdots, M_d\}$ we conclude that

$$N := \bigcup_{i=1}^{d} f_i(N_i) \supseteq q_0^{-1}(R'(F_3, H))^c,$$

where $f_i$ are defined in (15) and

$$q_0^{-1}(R'(F_3, H))^c \subset K^3$$

is the complement of $q_0^{-1}(R'(F_3, H))$ in $K^3$.

Therefore, proof of part (1) is complete. To prove part (2), we note that

$$\dim N_i = \dim K + 2 \cdot \dim M_i$$

for all $i \in [1, d]$. It was shown earlier that $\dim M_i \leq \dim K - 2$. Therefore,

$$\dim K^3 - \dim N_i = 3 \cdot \dim K - \dim N_i \geq 3 \cdot \dim K - 3 \cdot \dim K + 4 = 4.$$

This completes the proof of the lemma. □

**Proposition 3.2.** *Consider the $K/Z$-principal bundle $q_0^{-1}(R'(F_3, H)) \longrightarrow R'(F_3, H)$, where $q_0$ is the projection in (12). The induced homomorphism on fundamental groups*

$$\pi_1(K/Z) \longrightarrow \pi_1(q^{-1}(R'(F_3, H)))$$

*obtained from the homotopy exact sequence is trivial.*

*Proof.* Let $x_1, x_2, x_3 \in K$ be regular elements; we recall that $x \in K$ is called regular if the centralizer $C(x) = \{y \in K \mid yx = xy\}$ is a maximal torus in $K$. Since the set of regular elements is dense in $K$, and $q_0^{-1}(R'(F_3, H)) \subset K^3$ is a nonempty open dense subset (this follows from Lemma 3.1), we may choose these $x_i$, $i = 1, 2, 3$, to lie in $q_0^{-1}(R'(F_3, H))$.

Consider the orbit $\text{Orb}_{K^3}(x_1, x_2, x_3)$, of $(x_1, x_2, x_3) \in K^3$ for the adjoint action of the group $K^3$ on itself. Clearly we have the following identification of this orbit:

$$\text{Orb}_{K^3}(x_1, x_2, x_3) = (K/C(x_1)) \times (K/C(x_2)) \times (K/C(x_3))$$

with $C(x_i) \subset K$ being the centralizer of $x_i$. The conjugation action of $K$ on $\text{Hom}(F_3, K)$ coincides with the restriction of above action of $K^3$ to the image of the diagonal map $K \hookrightarrow K^3$. Therefore, the fibre $K/Z$ through the point $(x_1, x_2, x_3) \in q_0^{-1}(R'(F_3, H))$ of the $K/Z$-bundle

$$q_0^{-1}(R'(F_3, H)) \longrightarrow R'(F_3, H)$$

is contained in the orbit

$$(K/C(x_1)) \times (K/C(x_2)) \times (K/C(x_3)).$$

Now if we choose a point $(x_1, x_2, x_3) \in K^3$ which is general enough, then by the definition of the inverse image $q_0^{-1}(R'(F_3, H))$ and Lemma 3.1(2) it follows immediately that the complement of the open dense subset

$$q_0^{-1}(R'(F_3, H)) \bigcap \left( \frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)} \right) \subset \frac{K}{C(x_1)} \times \frac{K}{C(x_2)} \times \frac{K}{C(x_3)}$$

is of codimension at least four.

Since the image of $K/Z$ in $q_0^{-1}(R'(F_3, H))$ lies in $(q^{-1}(R'(F_3, H))) \bigcap \mathrm{Orb}_{K^3}(x_1, x_2, x_3)$, whose complement is of codimension at least four in

$$\mathrm{Orb}_{K^3}(x_1, x_2, x_3) = (K/C(x_1)) \times (K/C(x_2)) \times (K/C(x_3))$$

it follows that the homomorphism $\pi_1(K/Z) \longrightarrow \pi_1(q_0^{-1}(R'(F_3, H)))$ in the proposition factors through

$$\pi_1((q_0^{-1}(R'(F_3, H))) \cap \mathrm{Orb}_{K^3}(x_1, x_2, x_3))$$

$$= \pi_1((K/C(x_1)) \times (K/C(x_2)) \times (K/C(x_3))) = \pi_1(K/T)^3$$

where $T$ is a maximal torus of $K$.

For any maximal torus $T \subset K$, the quotient $K/T$ is diffeomorphic to $H/B$, where $B$ is a Borel subgroup of $H$. Since $H/B$ is simply connected, we conclude that $(K/C(x_1)) \times (K/C(x_2)) \times (K/C(x_3))$ is simply connected. This completes the proof of the proposition. □

*We now complete the proof of the non-existence of the covering $\widetilde{M} \longrightarrow q^{-1}(R'(F_3, H))$ as in* (13).

Since the homomorphism of fundamental groups $\pi_1(K/Z) \longrightarrow \pi_1(q_0^{-1}(R'(F_3, H)))$ is trivial (see Proposition 3.2), the induced covering $K \longrightarrow K/Z$ is trivial (see the diagram (13)). But this is a contradiction to the fact that $K$ is connected.

Therefore, we have proved the following theorem:

**Theorem 3.3.** *If the centre $Z(H)$ is non-trivial, then there is no universal bundle over $X \times \mathcal{M}'_X(H)$.*

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